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Statistical approach to brittle fracture

A. De S. JAYATILAKA, K. TRUSTRUM
School of Engineering and Applied Sciences, and School of Mathematical and Physical Sciences, Universtiy of Sussex, Brighton, UK

A general expression for the failure probability of a brittle material is developed by using the properties of flaw size distribution and the stress necessary to fracture an inclined crack. A comparison is made with Weibull analysis and an expression for the Weibull modulus, which is known to be an empirical material constant, is related to the properties of the flaw size distribution of a material. Limitations in the application of Weibull analysis are also discussed.

1. Introduction

Brittle materials exhibit a scatter of failure strengths unlike the ductile materials where plastic deformation takes place. The mode of fracture in a homogenous brittle material depends on the stress necessary to propagate an existing critical flaw or crack in it. In certain materials flaws can be inclusions, segregations or any other centres which give rise to incompatible deformations. Therefore variable sizes, shapes and orientations (with respect to the applied load) of the flaws in a material can account for the observed scatter of fracture strengths, when nominally identical specimens are tested under nominally identical loading conditions.

A statistical method commonly used to determine the strength of brittle materials is that given by Weibull [1]. In his theory an empirical formula of the form given below is used to relate the probability of failure, P_f, with stress, σ.

$$P_f = 1 - \exp\left\{-\int_V \left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m dV\right\}, \quad (1)$$

where m is a parameter (sometimes termed the Weibull modulus) determined experimentally, V is the volume of material, σ_o is a normalizing factor, and σ_u is the stress at which there is zero probability of failure. It is important to note that m which is a material characteristic has no real relationship to the micro- or macrostructure of the material. In Weibull analysis it is assumed that fracture at the most critical flaw under a given stress distribution leads to total failure. Thus, it is based on the idea of the "weakest link of a chain" concept as opposed to the parallel concept in which the failure of "one chain" causes redistribution of load among the other "chains", with total failure only taking place when the entire system is no longer capable of carrying the redistributed load.

In this paper a general expression for the failure probability of a brittle material is developed for a uniaxial tensile loading case by using the properties of flaw size distribution of a material and the tensile stress required to propagate a crack of a given size in a specific orientation to the applied load. This theory is then compared with Weibull analysis and an expression for the Weibull modulus, hitherto considered as an empirical constant, is related to the properties of the flaw size distribution of a material.

2. Theory

The stress necessary to propagate an inclined crack, as shown in Fig. 1, has been studied by Sih [2] and Jayatilaka *et al.* [3] using strain energy concepts. They showed that the initial crack growth occurs when the strain energy density of a body attains a minimum value. Using this concept, the strength, σ, of a brittle material is given by

$$\sigma^2 a = L(\beta, \nu) \quad (2)$$

where a is the semi-crack length, β is the crack angle and ν is the Poisson's ratio of the material. $L(\beta, \nu)$ is a function of β and ν. Equation 2 may be rewritten in an analytical form (see Appendix 1) for $\nu = 0.25$ as

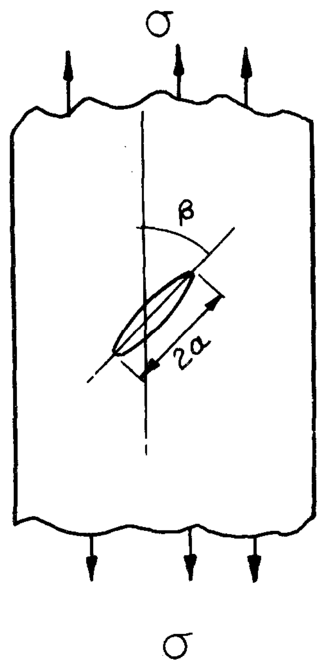

Figure 1 An inclined crack under a uniform tensile stress.

$$\sigma^2 a = \tfrac{1}{2} K_{IC}^2 \beta^{-1} \quad (3)$$

where K_{IC} is the critical stress intensity factor of the material.

Let $f(a)$ be the probability density of the semi-crack length, where $a \geqslant 0$. Assuming that any crack angle is equally likely, the probability density of β is $2/\pi$ for $0 \leqslant \beta \leqslant \pi/2$. The probability of failure, $F(\sigma)$, at a stress, σ, for one crack is given by

$$F(\sigma) = \int\int \frac{2}{\pi} f(a) \mathrm{d}a \mathrm{d}\beta \quad (4)$$

$$\frac{L(\beta, \nu)}{a} \leqslant \sigma^2$$

$$0 \leqslant \beta \leqslant \pi/2.$$

From Equations 2 and 3,

$$L(\beta, V) = \tfrac{1}{2} K_{IC}^2 \beta^{-1} \quad (5)$$

Experimental results by Poloniecki [4], and Poloniecki and Wilshaw [5] suggest that $f(a)$ can be closely fitted by the following expression (see Fig. 2).

$$f(a) = \frac{c^{n-1}}{(n-2)!} a^{-n} \mathrm{e}^{-c/a} \quad (6)$$

where c is a scaling parameter and n is the rate at

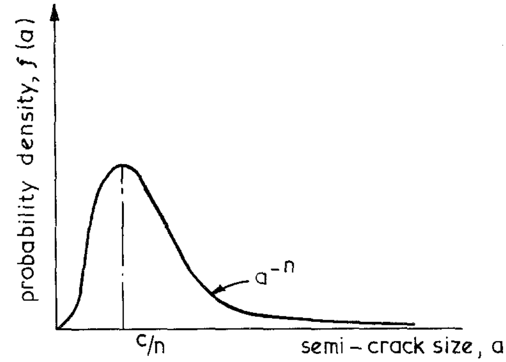

Figure 2 Distribution of crack size in a material where c/n is the mode.

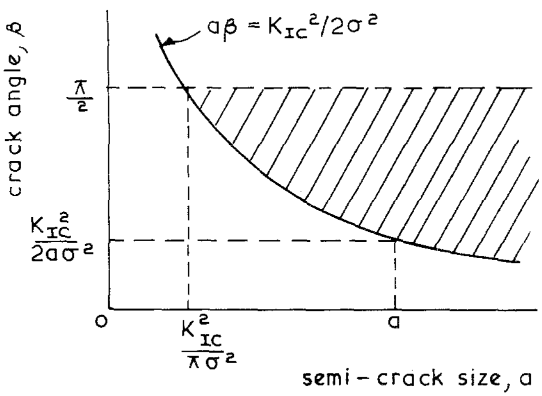

Figure 3 Constant stress curve showing limits of integration.

which the density tends to zero. It is important to note from Equation 3 that the strength is controlled by the flaws found in the "tail" of the curve. Thus, it follows that an error in the function to describe the crack size distribution for small a is not serious. Equation 4 now takes the form

$$F(\sigma) = \int\int \frac{2}{\pi} \frac{c^{n-1} a^{-n} \mathrm{e}^{-c/a}}{(n-2)!} \mathrm{d}a \mathrm{d}\beta \quad (7)$$

$$\frac{K_{IC}^2}{2\beta a} \leqslant \sigma^2$$

$$0 \leqslant \beta \leqslant \pi/2.$$

Fig. 3 shows the limits of integration. On substituting the limits, Equation 7 takes the form

$$F(\sigma) = \int_{K_{IC}^2/(\pi\sigma^2)}^{\infty} \left[\int_{K_{IC}^2/(2a\sigma^2)}^{\pi/2} \frac{2}{\pi} \frac{c^{n-1} a^{-n}}{(n-2)!} \mathrm{e}^{-c/a} \mathrm{d}\beta \right] \mathrm{d}a \quad (8)$$

and one integration gives

$$F(\sigma) = \int_{K_{IC}^2/(\pi\sigma^2)}^{\infty} \left[1 - \frac{K_{IC}^2}{\pi a \sigma^2} \right] \frac{c^{n-1} a^{-n} \mathrm{e}^{-c/a}}{(n-2)!} \mathrm{d}a. \quad (9)$$

For N cracks, the probability of failure, P_f, is given by

$$P_f = 1 - \text{probability of survival of all } N \text{ cracks}$$

and hence

$$P_f = 1 - [1 - F(\sigma)]^N. \quad (10)$$

Therefore P_f can be calculated for different values of σ and the mean stress, $\bar{\sigma}$, is given by

$$\bar{\sigma} = \int_0^1 \sigma \mathrm{d}P_f = \int_0^\infty (1 - P_f)\mathrm{d}\sigma. \quad (11)$$

3. Relation with Weibull distribution

3.1. Expression for m

The expression for $F(\sigma)$ given in Equation 9 can be found using the substitution, $u = c/a$, and for small values of σ $(\sigma\sqrt{(\pi c)}/K_{IC} \ll 1)$,

$$F(\sigma) = \frac{1}{n!}\left(\frac{\sigma^2 \pi}{K_{IC}^2}\right)^{n-1}\left[1 - \frac{\sigma^2 \pi c}{K_{IC}^2}\left(\frac{n-1}{n+1}\right) + 0\left(\frac{\sigma^4 \pi^2 c^2}{K_{IC}^4}\right)\right]. \quad (12)$$

When N is large, P_f, given by Equation 10, takes the form

$$P_f \simeq 1 - \exp[-NF(\sigma)]. \quad (13)$$

From Equations 12 and 13, it follows that for large N and small σ

$$P_f \simeq 1 - \exp\left[-N\frac{c^{n-1}}{n!}\left(\frac{\pi\sigma^2}{K_{IC}^2}\right)^{n-1}\right]. \quad (14)$$

The above expression may be written in the form

$$P_f = 1 - \exp[-k_1 N(\sigma)^{2n-2}] \quad (15)$$

where k_1 is a constant for a given material. In Weibull analysis, the expression given in Equation 1 takes the form, for a uniaxial loading case, when σ_u is assumed zero as for most brittle materials,

$$P_f = 1 - \exp\left[-V\left(\frac{\sigma}{\sigma_0}\right)^m\right]. \quad (16)$$

Since the number of cracks is proportional to the volume of material, the above expression can be rewritten as

$$P_f = 1 - \exp[-k_2 N(\sigma)^m]. \quad (17)$$

Hence the values of m and n are related by Equation 18. k_2 in Equation 17 is a constant for a given material.

$$m = 2n - 2 \quad (18)$$

By fitting the curve given by Equation 6 to the results of [5], the value of n for glass was estimated to be 2.67, with a standard error of 0.44, which gives $m = 3.34 \pm 0.88$.

3.2. Effect of number of cracks

In Weibull analysis the expression for mean strength, $\bar{\sigma}_w$, may be written [6] in the form,

$$\bar{\sigma}_w = \frac{\sigma_0}{V^{1/m}}\Gamma\left(1 + \frac{1}{m}\right) \quad (19)$$

Γ is the "gamma" function. Since the volume of material is proportional to the number of cracks, the above expression takes the form

$$\bar{\sigma}_w = k_3 N^{-1/m} \quad (20)$$

where k_3 is a constant for a given material. Therefore, Equation 20 may be written as

$$\log \bar{\sigma}_w = \log k_3 - \frac{1}{m}\log N. \quad (21)$$

Mean strengths, $\bar{\sigma}$, given by Equation 11, were evaluated by numerical integration for $n = 2, 3, 4$ and 6 (corresponding $m = 2$, 4, 6 and 10) and are plotted in Fig. 4 as a function of N. The corresponding lines, given by Equation 21 are drawn through the end points. Expressions for $F(\sigma)$ for each case are given in Appendix 2.

It is worth noting that changing the length scale c in the normalized mean stress, $\bar{\sigma}\sqrt{(\pi c)}/K_{IC}$, only alters the intercepts of the straight lines in Fig. 4, and even if c is replaced by the mode, c/n, of the assumed flaw size distribution, the lines still occur in the same order.

It is evident from the above results that there is a lowest limit for N for a given value of n or m for which the Weibull analysis is a good approximation. Defining a good approximation to be when the relative error $(\bar{\sigma} - \bar{\sigma}_w)/\bar{\sigma} \leqslant 0.05$, the lowest values of N, obtained by interpolation from the results displayed in Fig. 4, are given in Table I.

The gradients of the straight lines drawn through the two end points ($N = 600$ and 1000) in Fig. 4 are nearly equal to the "Weibull slope". These values are listed in Table II. Also shown in Table II

TABLE I Lowest values of N for which Weibull analysis is a good approximation

n	$m = 2n - 2$	N
2	2	3
3	4	28
4	6	60
6	10	110

TABLE II Comparison of "slopes" and distribution functions due to Weibull and authors

n	m	Negative "slope"		$\text{Max}\,\lvert P_{f,w} - P_f \rvert$
		Weibull	Authors	
2	2	0.500	0.503	0.001
3	4	0.250	0.255	0.004
4	6	0.167	0.178	0.014
6	10	0.100	0.118	0.035

is the maximum deviation between the probability of failure, $P_{f,w}$, calculated from the Weibull distribution Equations 16 and 19, and P_f calculated from Equation 10 with $N = 1000$.

4. Discussion

The theory described in this paper provides physical meaning to the empirical constant used in statistical methods to evaluate the strength of brittle materials. The Weibull modulus, m which is hitherto understood as an empirical constant can now be related to the properties of flaw size distribution in a material. For a given material, this distribution may be obtained by non-destructive methods, etc. For the assumed flaw size distribution (Fig. 2), the variability of crack size in a given volume is greater for the smaller values of n, and hence of m by Equation 18. This means that for materials with smaller values of m, a large crack is more likely to be present, and so the mean strength for a given volume is less. Consequently the theory predicts that materials with lower values of m are more brittle, which was also found by Mitchell [6], who listed the values of m for different materials.

The Weibull analysis is only a particular case of the theory described in this paper. It can be seen that even for a given material prepared under different fabrication methods, in order to produce either a change in the total number of cracks in the body or to change the crack size distribution, the theory described in this paper is easily applicable. If the number of cracks is reduced by decreasing the volume of a material, then it is evident from Fig. 4 that below a certain number of cracks the Weibull analysis cannot be applied. This particularly applies to materials with high m values where certain minimum number of cracks must be present (given in Table I) in order to apply Weibull analysis. In all such cases, the strengths may be evaluated using the general expression given in the text. Thus, the reasons for the limitations in the use of Weibull analysis to predict the strengths of brittle materials can be better understood.

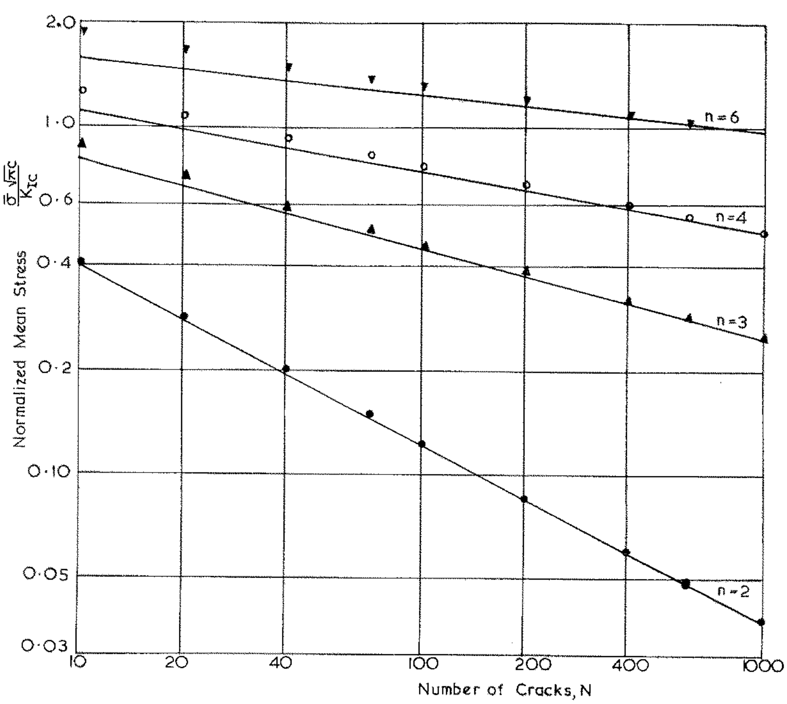

Figure 4 Logarithmic plot of normalized mean stress (points with each symbol representing a different value of n) in terms of number of cracks. Straight lines correspond to the Weibull "slopes".

Appendix 1. Relation between probability of failure P_f, and critical stress intensity factor, K_{IC}

From the work of Sih [2] and Jayatilaka *et al.* [3], the strength of a body containing an inclined crack can be related to the critical strain energy density factor, S_{cr}, which is a material property (constant) by

$$\sigma^2 a = L(\beta, \nu) = \frac{8ES_{cr}}{1+\nu} u(\beta, \nu) \quad \text{(A1)}$$

where E is the Young's modulus. For an opening mode (mode I) fracture, i.e. $\beta = \pi/2$, the stress to propagate a crack of a given size is minimum. Hence $u(\beta, \nu)$ reaches a minimum value given by U_{min}. Therefore the above equation becomes

$$\sigma_I{}^2 a = \frac{8ES_{cr}}{1+\nu} u_{min} \quad \text{(A2)}$$

where σ_I is the fracture stress in mode I. For mode I, the stress can also be related to the critical stress intensity factor, K_{IC}, by the following expression, which is commonly used in fracture toughness studies,

$$\sigma_I{}^2 a = \frac{K_{IC}^2}{\pi} \quad \text{(A3)}$$

for an infinite body. Therefore from Equations A1, A2 and A3 the following expression is obtained:

$$\sigma^2 a = K_{IC}^2 \frac{1}{\pi} u(\beta, \nu) \frac{1}{u_{min}}. \quad \text{(A4)}$$

Results given by Sih [2] and Jayatilaka *et al.* [3] for u_{min} and $u(\beta)$ for $\nu = 0.25$ (the value used for brittle materials) can be used to obtain an analytical expression for $u(\beta)$ by suitable curve fitting. For the value of ν considered, Equation A4 takes the form

$$\sigma^2 a = \tfrac{1}{2} K_{IC}^2 \beta^{-1}. \quad \text{(A5)}$$

Appendix 2. Expressions for $F(\sigma)$

Using the substitutions $x = \sigma^2 \pi c/K_{IC}^2$ and $u = c/a$, Equation 9 takes the form,

$$F(\sigma) = \int_0^x \left(1 - \frac{u}{x}\right) \frac{u^{n-2} e^{-u}}{(n-2)!} du. \quad \text{(A6)}$$

The above equation can now be integrated for different values of n.

$$\text{When } n = 2, F(\sigma) = 1 + \frac{1}{x}(e^{-x} - 1) \quad \text{(A7)}$$

$$n = 3, F(\sigma) = 1 + e^{-x} + \frac{2}{x}(e^{-x} - 1) \quad \text{(A8)}$$

$$n = 4, F(\sigma) = 1 + 2e^{-x} + \frac{x}{2}e^{-x} + \frac{3}{x}(e^{-x} - 1) \quad \text{(A9)}$$

$$n = 6, F(\sigma) = 1 + 4e^{-x} + \frac{3}{2}xe^{-x} + \frac{x^2}{3}e^{-x} + \frac{x^3}{24}e^{-x} + \frac{5}{x}(e^{-x} - 1) \quad \text{(A10)}$$

where

$$x = \sigma^2 \pi c/K_{IC}^2.$$

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